Kalman Filter

Let $(X_n)_{n \geq 0}$ evolve according to:

$$X_{n+1} = aX_n + V_n \qquad n \geq 0$$

$$Y_n = X_n + W \qquad n \geq 1$$

X_0, $(V_n)_{n\geq 0}$, $(W_n)_{n \geq 0}$ are all zero-mean, uncorrelated rv's.

$$Var(X_0) = \sigma_x^2$$

$$Var(V_n) = \sigma_v^2$$

$$Var(W_n) = \sigma_w^2$$

Scalar KF:

Initialize: $\hat{X}_{0|0} = 0$, $\sigma^2_{0|0} = \sigma_x^2$

For $n \geq 1$, do:

get a new estimator — from old estimator — using new observation — Kalman gain

(a) $$\hat{X}_{n|n} = a\hat{X}_{n-1|n-1} + \underbrace{K_n}_{\text{Kalman gain}}(\underbrace{Y_n}_{\text{observation}} - a\hat{X}_{n-1|n-1})$$

(b) $$\underbrace{\sigma^2_{n|n-1}}_{\text{error}} = a^2 \sigma_{n-1|n-1} + \sigma_v^2$$

(c) $$K_n = \frac{\sigma^2_{n|n-1}}{\sigma^2_{n|n-1} + \sigma_w^2}$$

compute this first then plug in above

(d) $$\underbrace{\sigma_{n|n}^2}_{\text{Mean squared error}} = (1 - K_n)\sigma^2_{n|n-1}$$

$$\boxed{\begin{aligned} \hat{X}_{n|m} &= \mathbb{E}[X_n \mid Y^m] \\ \sigma^2_{n|m} &= \mathbb{E}[X_n - \hat{X}_{n|m}]^2 \end{aligned}}$$

Proof of Correctness:

Strategy: Let $\tilde{Y}_1, \tilde{Y}_2 \ldots$ be the linear innovation sequence for our observations

ie: $\tilde{Y}_n = Y_n - \mathbb{L}[Y_n \mid Y^{n-1}]$

Big idea:

$$\hat{X}_{n|n} = \mathbb{L}[X_n \mid \tilde{Y}^n]$$

$$= \mathbb{L}[X_n \mid Y^{n-1}] + \mathbb{L}[X_n \mid \tilde{Y}_n]$$

$$= \mathbb{L}[aX_{n-1} + V_{n-1} \mid Y^{n-1}] + \frac{cov(X_n, \tilde{Y}_n)}{Var(\tilde{Y}_n)}\tilde{Y}_n$$

since projection is a linear operator, we get that

$$= a\mathbb{L}[X_{n-1} \mid Y^{n-1}] + \underbrace{\cancel{\mathbb{L}[V_{n-1} \mid Y^{n-1}]}}_{0 \text{ bc uncorrelated (by state-space model)}} + \frac{cov(X_n, \tilde{Y}_n)}{Var(\tilde{Y}_n)}\tilde{Y}_n$$

$$= a\hat{X}_{n-1|n-1} + \frac{cov(X_n, \tilde{Y}_n)\tilde{Y}_n}{Var(\tilde{Y}_n)}$$

$$\hat{X}_{n|n} = a\hat{X}_{n-1|n-1} + \underbrace{\frac{Cov(X_n, \tilde{Y}_n)}{Var(\tilde{Y}_n)}}_{=:K_n} \tilde{Y}_n$$

$$= a\hat{X}_{n-1|n-1} + K_n \tilde{Y}_n = a\hat{X}_{n-1|n-1} + K_n(Y_n - a\hat{X}_{n-1|n-1})$$ ⓐ ✓

$$\tilde{Y}_n = Y_n - \mathbb{L}[Y_n \mid Y^{n-1}]$$

$$= Y_n - \mathbb{L}[X_n + W_n \mid Y^{n-1}]$$

$$= Y_n - \mathbb{L}[aX_{n-1} + V_{n-1} + W_n \mid Y^{n-1}]$$

$$= Y_n - a\hat{X}_{n-1|n-1} + 0 + 0$$

For ⓑ:

$$\sigma^2_{n|n-1} = \mathbb{E}[\hat{X}_{n|n-1} - X_n]^2$$

$$= \mathbb{E}\Big(\mathbb{L}[\underbrace{aX_{n-1} + \cancelto{0}{V_{n-1}}}_{X_n} \mid Y^{n-1}] - (\underbrace{aX_{n-1} + V_{n-1}}_{X_n})\Big)^2$$

$$= \mathbb{E}[a\hat{X}_{n-1|n-1} - aX_{n-1} - V_{n-1}]^2$$

$$= a^2\sigma^2_{n-1|n-1} + \sigma_V^2 - \cancelto{0}{2a\,\mathbb{E}[(\hat{X}_{n-1} - X_{n-1})V_n]}$$

$$= a^2\sigma^2_{n-1|n-1} + \sigma_V^2$$ ⓑ ✓

For ⓒ:

$$K_n = \frac{Cov(X_n, \tilde{Y}_n)}{Var(\tilde{Y}_n)}$$

$$= \frac{\mathbb{E}[X_n(Y_n - \mathbb{L}[Y_n \mid Y^{n-1}])]}{Var(\tilde{Y}_n)}$$

$$= \frac{\mathbb{E}[X_n(X_n + \cancel{W_n} - \mathbb{L}[X_n + \cancel{W_n} \mid Y^{n-1}])]}{Var(\tilde{Y}_n)}$$

orthogonality principle

E[this] = 0 bc it's orthogonal

$$= \frac{\mathbb{E}[(X_n - \mathbb{L}[X_n \mid Y^{n-1}])(X_n - \mathbb{L}[X_n \mid Y^{n-1}])]}{Var(\tilde{Y}_n)}$$

$$= \frac{\sigma^2_{n|n-1}}{Var(\tilde{Y}_n)}$$

$$= \frac{\sigma^2_{n|n-1}}{\mathbb{E}[(Y_n - \mathbb{L}[Y_n \mid Y^{n-1}])\tilde{Y}_n]}$$

as before

$$= \frac{\sigma^2_{n|n-1}}{\mathbb{E}[(X_n + W_n - \hat{X}_{n|n-1})]\tilde{Y}_n]}$$

$$= \frac{\sigma^2_{n|n-1}}{\mathbb{E}[(X_n - \hat{X}_{n|n-1} + W_n)(X_n - \hat{X}_{n|n-1} + W_n)]}$$

$$= \frac{\sigma^2_{n|n-1}}{\sigma^2_{n|n-1} + \sigma^2_w + \underbrace{0}_{\text{cross terms}}}$$

$$= \frac{\sigma^2_{n|n-1}}{\sigma^2_{n|n-1} + \sigma^2_w} \quad ⓒ \ \checkmark$$

For ⓓ:

$$\sigma^2_{n|n} = \mathbb{E}[\hat{X}_{n|n} - X]^2$$

$$= \mathbb{E}\left[\hat{X}_{n|n-1} + \frac{\operatorname{Cov}(X_n, \tilde{Y}_n)}{\operatorname{Var}(\tilde{Y}_n)}\tilde{Y}_n - X_n\right]^2$$

$$= \mathbb{E}\left[\hat{X}_{n|n-1} - X_n + \frac{\operatorname{Cov}(X_n, \tilde{Y}_n)}{\operatorname{Var}(\tilde{Y}_n)}\tilde{Y}_n\right]^2$$

$$= \sigma^2_{n|n-1} + \frac{\operatorname{Cov}(X_n, \tilde{Y}_n)^2}{\operatorname{Var}(\tilde{Y}_n)^2}\cancel{\operatorname{Var}(\tilde{Y}_n)} + \mathbb{E}[(\hat{X}_{n|n-1} - X_n)\tilde{Y}_n]\frac{\operatorname{Cov}(X_n, \tilde{Y}_n)}{\operatorname{Var}(\tilde{Y}_n)}$$

$$= \sigma^2_{n|n-1} + k_n \operatorname{Cov}(X_n, \tilde{Y}_n) + 2k_n \mathbb{E}[(\hat{X}_{n|n-1} - X_n)\tilde{Y}_n]$$

$$= \sigma^2_{n|n-1} + k_n \operatorname{Cov}(X_n, \tilde{Y}_n) + 2k_n \mathbb{E}[(\cancel{\hat{X}_{n|n-1}} - X_n)(\underbrace{Y_n - \mathbb{L}[Y_n | Y^{n-1}]}_{\text{orthogonal to all linear fcns of } (1, Y_1, \dots, Y_{n-1})})]$$

$$= \sigma^2_{n|n-1} + k_n \operatorname{Cov}(X_n, \tilde{Y}_n) - 2k_n \operatorname{Cov}(X_n, \tilde{Y}_n)$$

$$= \sigma^2_{n|n-1} + k_n \sigma^2_{n|n-1} - 2k_n \sigma^2_{n|n-1}$$

$$= (1 - k_n)\sigma^2_{n|n-1} \quad ⓓ \ \checkmark$$

Another typical model for a process (linear estimation):

A process $(X_n)_{n \in \mathbb{Z}}$ is wide-sense stationary ↗ good model for things in steady-state

if $\mathbb{E}[X_n] = \mathbb{E}[X_0]$ (its mean isn't changing in time)

& $\mathbb{E}[X_{n+k}X_k] = \mathbb{E}[X_n X_0]$ (covariance invariant to shifts in time)

$\Big\}\ \forall n, k \in \mathbb{Z}$

Time-Series Analysis

$$\left.\begin{array}{ll} \text{Let } (X_n)_{n\in\mathbb{Z}} & = \text{target process of interest} \\ \phantom{\text{Let }} (Y_n)_{n\in\mathbb{Z}} & = \text{observation process} \end{array}\right\} \text{both WSS}$$

Q/ How do we design a filter (ie, a linear system) H s.t.

$$(Y_n)_{n\in\mathbb{Z}} \longrightarrow \boxed{\quad H \quad} \longrightarrow (\hat{X}_n)_{n\in\mathbb{Z}}$$

, where $\hat{X}_n$ is best linear estimator of X_n given, say, $(Y_{n-k})_{k\geq 0}$

(causal estimator)